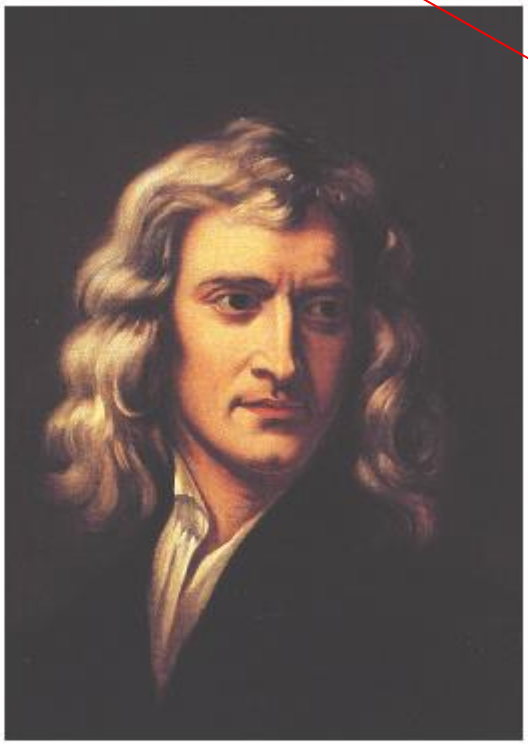
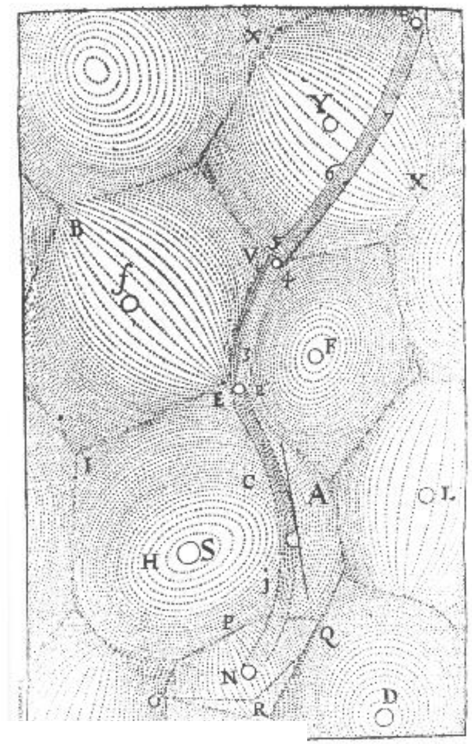
2.10. Gravitation

Newton Gave attention to the ',World" of R. Descartes (1596-1650)



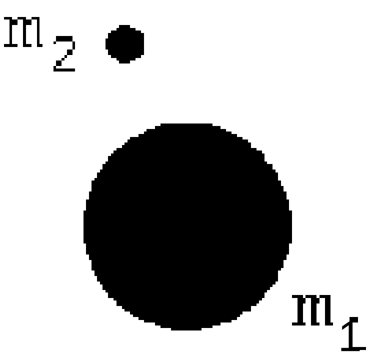
Sun and planets: Vortices
in a fluid



a) Law of gravitation
(Newton 1665)
All masses attract
each other!

F=F(m1, m 2, r,..) ; r
distance between m1, m2

How looks the form of the
function of attraction?



From observations:

1) Surface of the earth: g= konstant, F~ m2

2) Principle of reaction: $\vec{F}_{1/2} = -\vec{F}_{2/1} \rightarrow F \sim m_1$

Assumption.: Newton: Planet circels around the sun : One circle: $T = \frac{2\pi R}{v}$
with v on a circular orbit

Newton knew the third law of Kepler for orbits of the planets $R^3 \sim T^2$

using $T \approx \frac{R}{v}$ or $T^2 \approx \frac{R^2}{v^2}$

$R^3 \approx \frac{R^2}{v^2}$ or $R^2 \approx \frac{R}{v^2}$ or $\frac{1}{R^2} \approx \frac{v^2}{R}$

By other considerartions Newton found

The centrifugal force as $F_c \approx \frac{v^2}{R}$.

He concluded that: $F_c \approx \frac{1}{R^2}$ In equilibrium between m_1 and m_2

Centripetal force(Gravitation): $\approx \frac{1}{R^2}$

and found:Law of gravitation

$$F = G\frac{m_1 \cdot m_2}{R^2}$$

with G as constant of gravity!

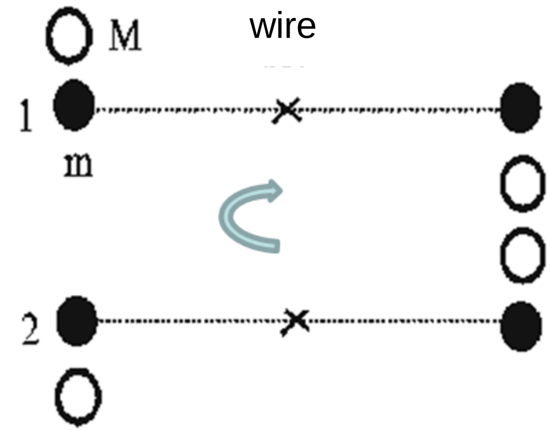
b) Mesurement of G wit torsion balance:

One of the richest men of that time

Measured by Henry Cavendish, published 1798

Great experiment!

wire

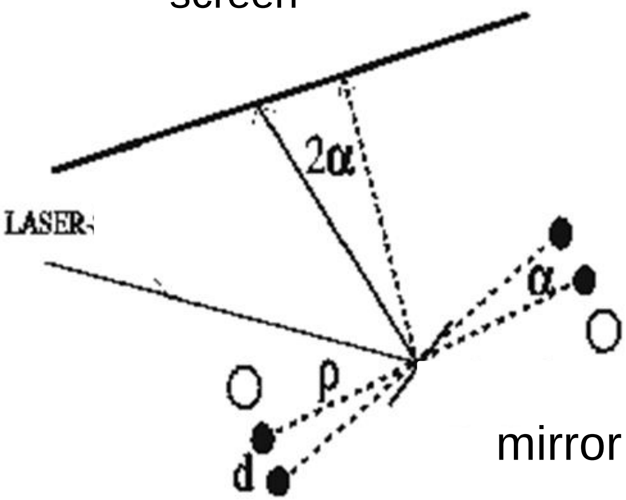


Wire with mirror (perpendicularl to image plane)

M=1.5kg, m=15g, L=26m, r=5cm,

Torsion balance so sensitiv, leading to a balanced condition which with Gravitation between the sheres results to unequal 0 .

screen



mirror

With t, test time, small compared to time of oscillation T of torsion balance and therefore approximations can be made.

$$m \cdot a = \frac{2 \cdot G \cdot m \cdot M}{r^2} \qquad d = 0.5 \cdot a \cdot t^2$$

$$\rightarrow a = \frac{2 \cdot d}{t^2} \qquad \alpha = \frac{d}{\rho} \qquad 2\alpha = \frac{S}{L} \qquad d = \frac{S \cdot \rho}{2 \cdot L}$$

$$G = \frac{a \cdot r^2}{2 \cdot M} \Longrightarrow G = \frac{r^2 \cdot S \cdot \rho}{t^2 \cdot M \cdot 2 \cdot L}$$

With results of a previous experiment in a empty lecture hall

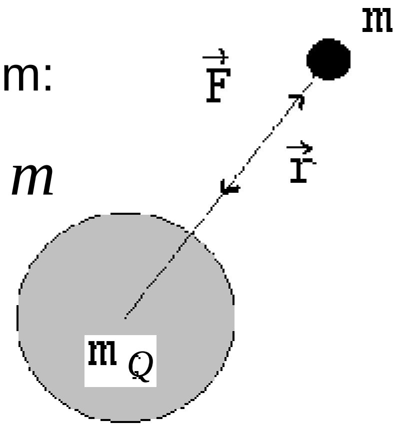
with t=60s

$$G = \frac{0.05 \cdot 0.05 \cdot 0.15 \cdot 0.05}{60 \cdot 60 \cdot 1.5 \cdot 2 \cdot 26} = 6.68 \cdot 10^{-11}\ m^3 \cdot s^{-2} \cdot kg^{-1}$$

c) Field of gravitation

Force of m_Q on m:

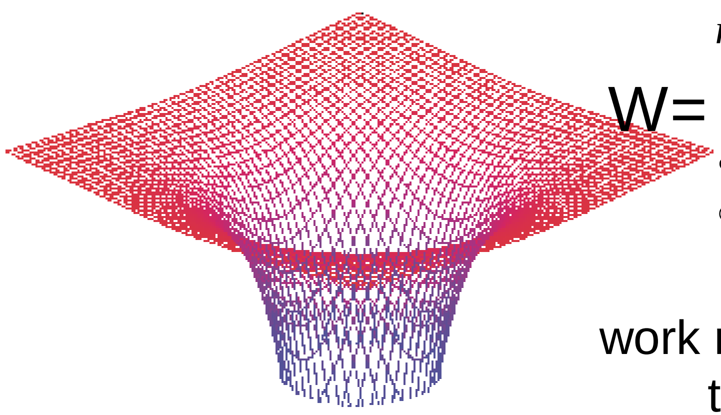
Field of gravitation $-\frac{G \cdot m_Q}{r^2} \cdot \vec{e}_r$ $\quad \vec{F} = -\frac{G \cdot m_Q}{r^2} \cdot \vec{e}_r \cdot m$



Mass m_Q creates a field of force,
the mass m „feels“ the force F

$$\vec{F} = \vec{g}(r) \cdot m$$

Moving m from r= $\infty \rightarrow r_0$ Gravitation supplies work.



$$W = \int_{\infty}^{r_0} \vec{F} dr = -G \cdot m_Q \cdot m \cdot \int_{\infty}^{r_0} \frac{dr}{r^2} = G \cdot m_Q \cdot \frac{m}{r_0}$$

reverse: $r_0 \rightarrow \infty$

work muß to be supplied against
the force of gravitation
This supplied force stays as potential energy
of m in the field of gravitation

$$f(x,y) = \frac{-1}{(x^2+y^2)}$$

$$-\frac{G \cdot m_Q}{r}$$

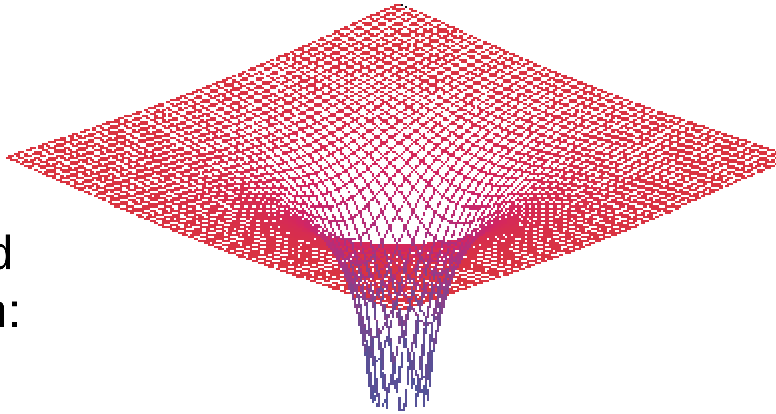
$$E_{pot}(r) = -\frac{G \cdot m_Q \cdot m}{r}$$

Potential

e.g.: m moves in the potential of mQ

$$f(x,y) = -\frac{1}{\sqrt{(x^2+y^2)}}$$

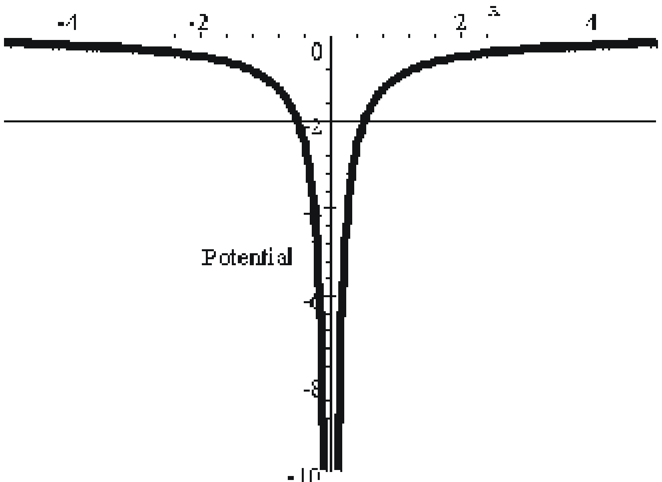
Fixing origin: $E_{pot}(r = \infty) = 0$



gravitation stands as an example of a conservative force.

Going around a closed path:

$$\oint \vec{F} \cdot d\vec{s} = 0$$



$$f(x) = \frac{-1}{\sqrt{x^2}}$$

Change of potentiel energy:

$$\Delta U = U_E - U_A = \int_E^A \vec{F} \cdot d\vec{r} = -\int_E^A \frac{G m_Q m}{r^2} dr$$

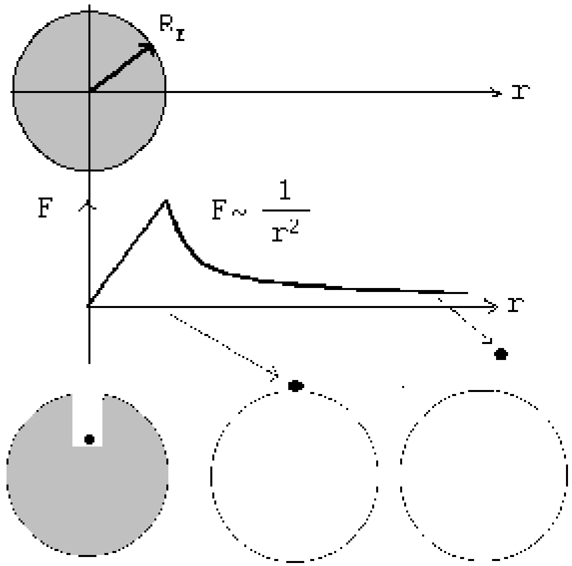
$$F(r) = -\frac{dU}{dr}$$

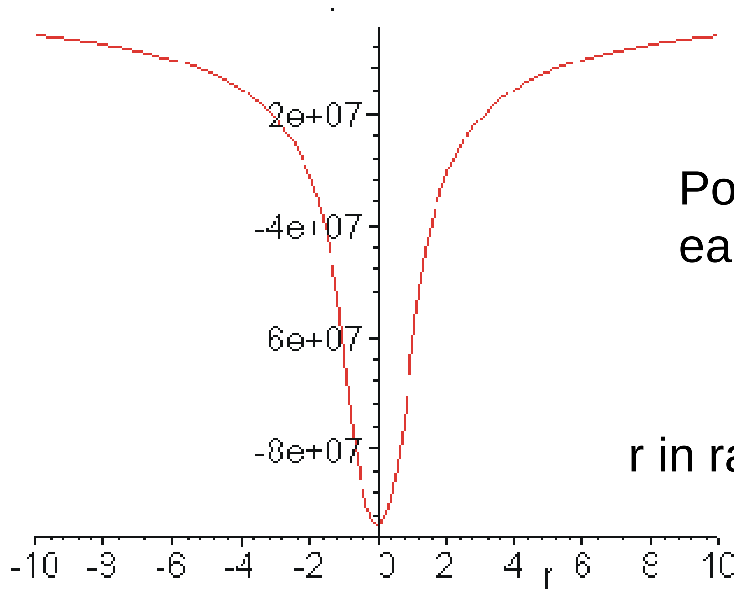
or in the form of a vector :

$$\vec{F}(\vec{r}) = -\vec{\nabla} \cdot U(\vec{r})$$

That means:: From a potential → Kraftfeld

(Model for the earth)

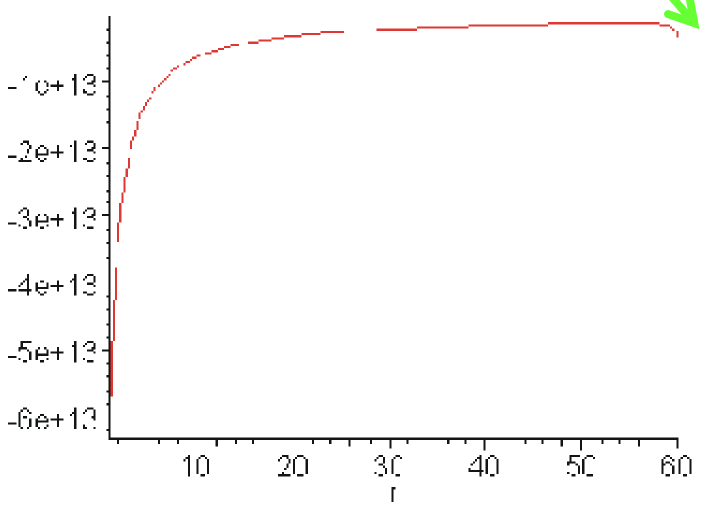


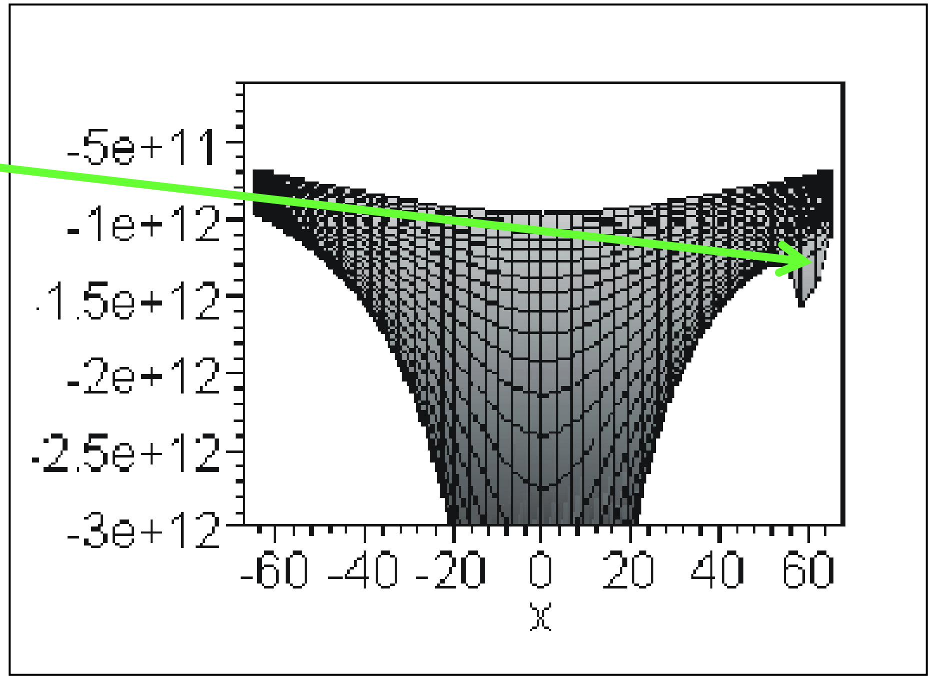


Potential of earth

r in radii of the earth

Where is the moon?





d) The laws of Kepler for movements of the planets

1. Law

The planets move on ellipses,
the sun occupies one of the foci

2. Law

Area rule: The vector sweepts through in equal times equal areas

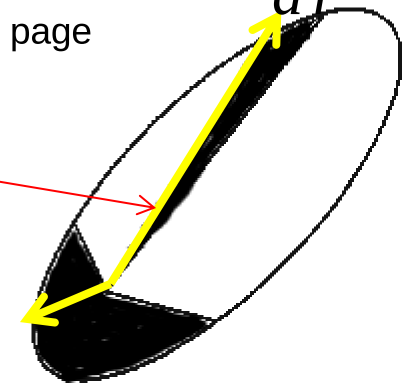
$\vec{A}$

μ

See next page

$d\vec{r}$

$$\vec{A} = \frac{1}{2}\left[\vec{r} \times \frac{d\vec{r}}{dt}\right] = \frac{\vec{L}}{2\mu}$$



constant velocity of areas
<->conservation of angular momentum

3. Law

The squares of orbital periods behave like third power of the semi-major axes

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

System with two particles with inner forces

$$m_1 \cdot \ddot{\vec{r}}_1 = \vec{F}_{21}, m_2 \cdot \ddot{\vec{r}}_2 = \vec{F}_{12} = -\vec{F}_{21} \qquad m_1 \cdot \ddot{\vec{r}}_1 + m_2 \cdot \ddot{\vec{r}}_2 = 0$$

What does c.m. doing?

$$\vec{r}_s = \frac{m_1 \cdot \vec{r}_1 + m_2 \cdot \vec{r}_2}{m_1 + m_2} \rightarrow \ddot{\vec{r}}_s = 0$$

Center of mass moves steady!

Where is the dynamics? In relative movements: $\vec{r} = \vec{r}_1 - \vec{r}_2$

To start the calculation: $\vec{r}_1, \vec{r}_2$ Replacingg by

$$\vec{r}_1 = \vec{r}_s + \frac{m_2}{m_1 + m_2} \vec{r}$$

With $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\vec{r}_2 = \vec{r}_s - \frac{m_1}{m_1 + m_2} \vec{r}$$

Behaves like a particle with mass : μ

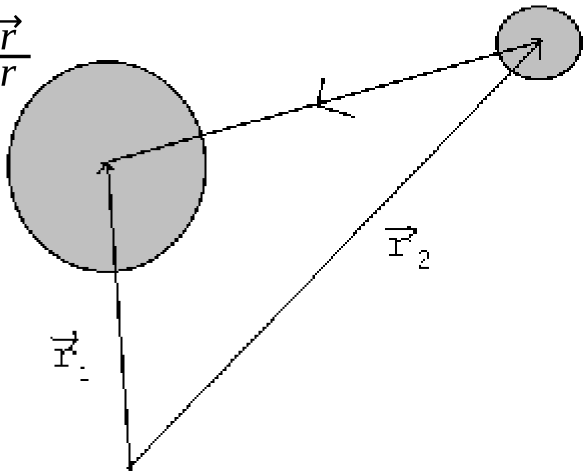
$$\mu \cdot \ddot{\vec{r}} = \vec{F}_{21}$$

Gravitation between two heavenly bodies:

$$m_1 \cdot \ddot{\vec{r}}_1 = -G \cdot \frac{m_1 \cdot m_2}{r^2} \frac{\vec{r}_1 - \vec{r}_2}{r} = -G \cdot \frac{m_1 \cdot m_2}{r^2} \frac{\vec{r}}{r}$$

$$m_2 \cdot \ddot{\vec{r}}_2 = -G \cdot \frac{m_1 \cdot m_2}{r^2} \frac{\vec{r}_2 - \vec{r}_1}{r} = G \cdot \frac{m_1 \cdot m_2}{r^2} \frac{\vec{r}}{r}$$

or $$\mu \cdot \ddot{\vec{r}} = -G \cdot \frac{m_1 \cdot m_2}{r^2} \frac{\vec{r}}{r}$$



Central force: $\vec{F}(r) = f(r) \cdot \vec{r}$

Angular momentum: $\vec{L} = \mu \cdot \vec{r} \times \dot{\vec{r}}$ Conserved movement, plane $\perp$ $\vec{L}$

Polar coordinates: $x(t) = r(t) \cdot \cos(\Phi(t)),\ y(t) = r(t) \cdot \sin(\Phi(t))$

$L_x = L_y = 0 \quad \hat{e}_z \cdot L_z = \mu \cdot r^2 \cdot \dot{\vec{\Phi}} = \overrightarrow{constant}$

$$\dot{\vec{\Phi}} = \frac{L}{\mu r^2}$$

Energy conservation:

$$E = \frac{1}{2}\mu \cdot v^2 + U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2 \dot{\Phi}^2) + U(r) = \text{constant}$$

solved $$\dot{r}^2 \rightarrow \frac{2(E-U)}{\mu} - r^2 \cdot \dot{\Phi}^2 = \dot{r}^2$$

"Division" with $\dot{\Phi} = \frac{L}{\mu \cdot r^2}$ $\quad \dot{r} = \sqrt{\frac{2(E-U)}{\mu} - \frac{L^2}{\mu^2 \cdot r^2}}$

$\frac{\dot{r}}{\dot{\Phi} \cdot r^2} = \sqrt{\frac{2(E-U)\cdot\mu}{L^2} - \frac{1}{r^2}}$ with $\frac{\frac{dr}{dt}}{\frac{d\Phi}{dt}} = \frac{dr}{d\Phi} \rightarrow \frac{1}{r^2}\frac{dr}{d\Phi} = \sqrt{\frac{2(E-U)\cdot\mu}{L^2} - \frac{1}{r^2}}$

Trick, to come to orbit

with $U(r) = \frac{-A}{r}$ *und* $\sigma(\Phi) = \frac{1}{r(\Phi)}$ $\rightarrow \frac{d\sigma}{d\Phi} = -\frac{1}{r^2}\frac{dr}{d\Phi}$

$-\frac{d\sigma}{d\Phi} = \sqrt{\frac{2\mu(E+A\cdot\sigma)}{L^2} - \sigma^2}$ with $p = \frac{L^2}{A\cdot\mu}$ $\quad \varepsilon = \sqrt{1 + \frac{2E\cdot L}{\mu\cdot A^2}}$

$(d\sigma/d\Phi)^2 + (\sigma - \frac{1}{p})^2 = \frac{\varepsilon^2}{p^2}$ Will be solved: $r(\Phi) = \frac{p}{1+\varepsilon\cdot\cos(\Phi-\Phi_0)}$

ansatz: $x = r \cdot \cos(\Phi) + c \quad y = r \cdot \sin(\Phi)$ and get

$r^2 = (x-c)^2 + y^2 = [p - \varepsilon \cdot r \cdot \cos(\Phi)]^2$ with $c = \frac{\varepsilon p}{1-\varepsilon^2}$

and a $a = \frac{p}{1-\varepsilon^2}$ $\rightarrow$

$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$ 1.Case $\varepsilon > 1$ e.g.: c> a :Hyperbola aus E>0

2.Case $\varepsilon < 1$ e.g.: c< a Ellipse aus E<0