2. Mechanics of a rigid body

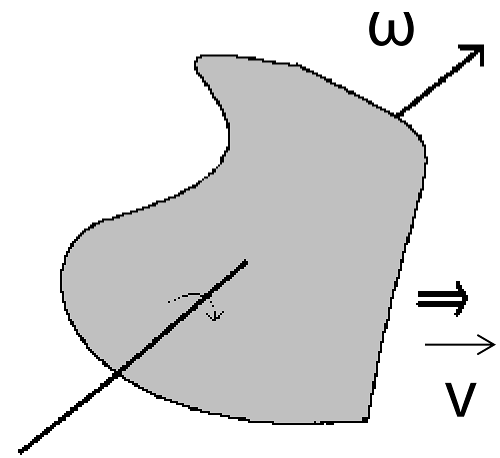
2.1. Translation and Rotation

In chapter 1: Mass point

$$\rightarrow\rightarrow\rightarrow\ \vec{r}(t)$$

In chapter 2: Mass points

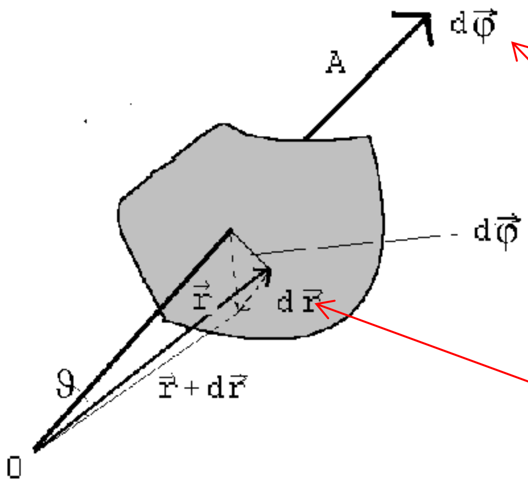
$$\rightarrow\rightarrow\rightarrow\ \sum_i \vec{r}_i(t)$$



Movement of a rigid body:

Rotation + Translation

Kinematics of rotation:



A: Axis of rotation

$\vec{\varphi}$: Axial Vektor (In direction of axis)

$$d\vec{r} = d\vec{\varphi} \times \vec{r} \qquad dr = d\varphi \bullet r \bullet \sin\vartheta$$

Velocity:

$$\boxed{\frac{d\vec{r}}{dt} = \vec{v} = \frac{d\vec{\varphi}}{dt} \times \vec{r} = \vec{\omega} \times \vec{r}}$$

In general for a point on a rigid body:

$$\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{r}$$

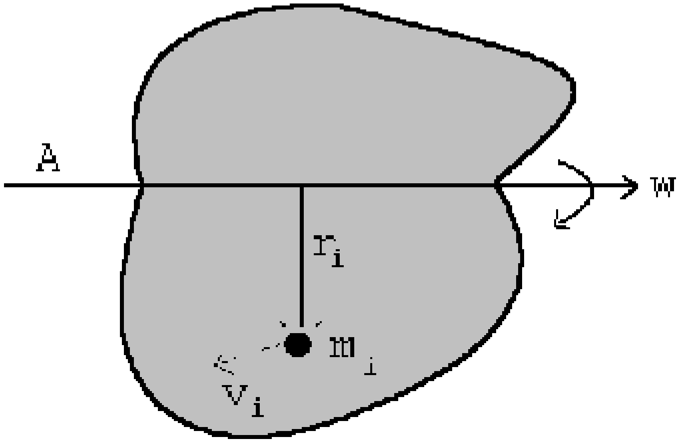
e.g. Translation + Rotation

Analoga:

	Translation	Rotation
Position	$\vec{r}$	$\vec{\varphi}$
Velocity	$\vec{v} = \dot{\vec{r}}$	$\vec{\omega} = \dot{\vec{\varphi}}$
Acceleration	$\vec{a} = \ddot{\vec{r}}$	$\dot{\vec{\omega}} = \ddot{\vec{\varphi}}$

2.2. Energy of rotation and moment of inertia

Mass point m_i with a distance r_i relativ to axis of rotation A, velocity v_i



and kinetic energy $\boxed{\frac{1}{2} m_i \cdot v_i^2}$

$$v_i = \omega_i \cdot r_i \quad \boxed{\rightarrow\rightarrow E_{kin_i} = \frac{1}{2}\omega^2 \cdot m_i \cdot r_i^2}$$

Total kinetic energy

Energy of rotation: $W_{Rot} = \frac{1}{2}\omega^2 \cdot \Sigma_i m_i \cdot r_i^2$

Transition to a continuous distribution of mass: $[\rho] = \frac{kg}{m^3}$

$$m_i \rightarrow \Delta m_i \rightarrow dm = \rho \cdot dV$$

$$\int dm \cdot r^2 \ : \ W_{Rot} = \frac{1}{2}\omega^2 \int dm \cdot r^2;$$

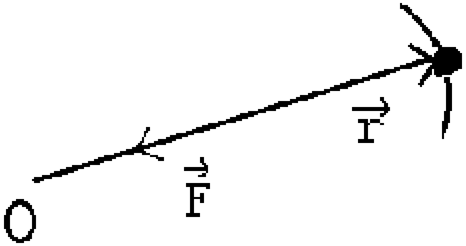
(Element of mass = density x element of volume)

Moment of inertia I: $[I] = kg \cdot m^2$

Analogy: Translation – Rotation: v-> ω,m-> I

I plays the same role of a movement of rotation around an axis, as m at a translation

2.3. Momentum of rotation and torque



Equation of movement: $\vec{F} = m \cdot \ddot{\vec{r}}$

Product of vectors $\vec{r} \rightarrow$

$\vec{r} \times \vec{F} = \vec{r} \times m \cdot \ddot{\vec{r}}$ We consider : Central force: $\vec{F} \parallel \vec{r} \rightarrow \vec{r} \times \vec{F} = 0$

$\rightarrow\rightarrow\rightarrow \vec{r} \times m \cdot \ddot{\vec{r}} = 0$

We recognize a new quantity: $\vec{L} = \vec{r} \times m \cdot \dot{\vec{r}}$

cause: $\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times m \cdot \dot{\vec{r}}\right) = \vec{r} \times m \cdot \ddot{\vec{r}} + \dot{\vec{r}} \times m \cdot \dot{\vec{r}}$

But: $\vec{r} \times m \cdot \ddot{\vec{r}}\ (s.o.) = 0 = \dot{\vec{r}} \times m \cdot \dot{\vec{r}}$ Momentum of rotation L constant over time

$\rightarrow\rightarrow \vec{L}$ Definition: $\boxed{\vec{L} = \vec{r} \times \vec{p}}$

At a rigid body: $\vec{L} = \sum_i m_i \cdot \vec{r}_i \times \vec{v}_i = \sum_i m_i \cdot \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$

Rules of calculations with vectors: $\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b})}$

$$\rightarrow\rightarrow \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \vec{\omega}(\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i \cdot (\vec{r}_i \cdot \vec{\omega})$$

Momentum of rotation of a rigid body: with $(\vec{r}_i \cdot \vec{r}_i) = r_i^2$

$$\vec{L} = \vec{\omega} \cdot \sum_i m_i \cdot r_i^2 = I \cdot \vec{\omega}$$

$\vec{r}_i \perp \vec{\omega}$ -> $(\vec{r}_i \cdot \vec{\omega})$ =0

$\frac{d\vec{L}}{dt}$??

$$[\vec{L}] = kg \cdot m^2 \cdot s^{-1}$$

$\vec{dL}/dt = \sum_i m_i \cdot \dot{\vec{r}}_i \times \vec{v}_i + \sum_i m_i \cdot \vec{r}_i \times \dot{\vec{v}}_i =$

Torque

$$\vec{T} = \frac{d\vec{L}}{dt}$$

Force times distance $\sum_i m_i \cdot \vec{r}_i \times \vec{a}_i$

$$\boxed{\frac{d\vec{L}}{dt} = \frac{d}{dt}(I \cdot \vec{\omega}) = I \cdot \dot{\vec{\omega}}}$$

Basic law of dynamics of movement of rotation

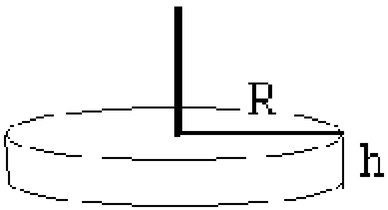
$\vec{T} = \dot{\vec{L}} \rightarrow\rightarrow\rightarrow \vec{T}$ causes change of $\vec{L}$

For a closed system: $\vec{T} = 0; \rightarrow\rightarrow\rightarrow\rightarrow \dot{\vec{L}} = 0$

The whole momentum of rotation of a system remains constant, as long as no torque from outside tackles.

Examples for momenta of inertia :

$$I = \int dm \cdot r^2 = \rho \cdot \int r^2 \cdot dV; mit \quad dV = 2\pi \cdot r \cdot dr \cdot dh$$



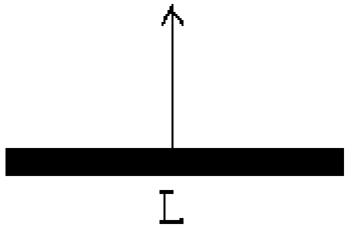
$$I = 2\pi \cdot \varrho \cdot h \cdot \int_0^R r^3 \cdot dr = 2\pi \cdot \varrho \cdot h \cdot \frac{r^4}{4} \Big|_0^R$$

$$I = \pi \cdot \varrho \cdot h \cdot \frac{R^4}{2}; da \quad M = \rho \cdot V = \pi \cdot R^2 \cdot h \cdot \rho$$

$$\Rightarrow \boxed{I = 0.5 \cdot M \cdot R^2}$$

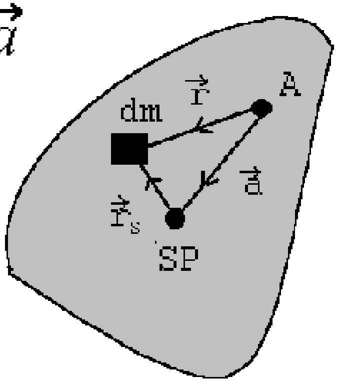
Similiar calculations: I(Sphere)=2*M*R^2 */5

I(Stick)=M*L^2/12



What happens, if the axis of rotation A not crosses the center of mass SP ?

$$\vec{r} = \vec{r}_s + \vec{a}$$



$$I = \int dm \bullet r^2 = \int dm \bullet \left(\vec{r}_s + \vec{a}\right)^2$$

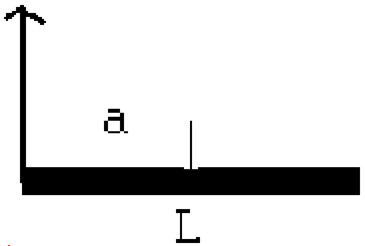
$$= \int dm \bullet r_s^2 + 2 \bullet \vec{a} \bullet \int dm \bullet \vec{r}_s + a^2 \int dm$$

$$= \int dm \bullet r_s^2 + 2 \bullet \vec{a} \bullet \int dm \bullet \vec{r}_s + a^2 \int dm$$

I = I_{SP} + 0 + a^2M

Rule of Steiner

Example, again stick:



Moment of inertia around any axis !

$$I = \frac{1}{12} M \bullet L^2 + M \bullet \frac{L^2}{4}$$

$$I = \frac{1}{3} M \bullet L^2$$

2.4. Conversion of energy

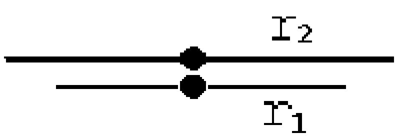
Analogy from :

Energy of rotation: $E_{kin_i} = \frac{m_i}{2} \cdot v_i^2$ $\sum \frac{m_i}{2} r_i^2 \cdot \omega^2$

leads to $\frac{1}{2} I \bullet \omega^2$

Force against a centrifugal force with diminishing r

Energy in a system of rotation, e.g. rotating stool



I gets smaller $\vec{L} = \vec{\omega} \cdot I$ r2→ r1 ω1 >ω2

$\Rightarrow E_{rot_1} \succ E_{rot_2}$

Energy of rotation? $E_{rot} = \frac{1}{2} I \bullet \omega^2$

because $L_1 = L_2$

Energy goes into the rotation system

$E_{rot} = \frac{1}{2} \frac{L^2}{I}$

2.5. Comparison: Translation- Rotation

Translation	Rotation
Distance: $\vec{s}(\vec{r})$	Angle $\vec{\phi}$
Velocity $\vec{v} = \dot{\vec{r}}$	Angular velocity $\vec{\omega} = \dot{\vec{\phi}}$
Acceleration $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$	Angular acceleration $\dot{\vec{\omega}} = \ddot{\vec{\phi}}$
Mass: m	Moment of inertia $I = \int dm \cdot r^2$
Force $\vec{F}$	Torque $\vec{T} = \vec{r} \times \vec{F}$

The basic laws

$\vec{F} = m \cdot \vec{a}$	$\vec{T} = I \cdot \dot{\vec{\omega}}$
$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{T} = \frac{d\vec{T}}{dt}$

Uniform movements $\vec{a} = 0, \vec{s} = \vec{v} \cdot t$ $\vec{\omega} = 0, \vec{\phi} = \vec{\omega} \cdot t$

Uniform accelerated movements:

$\vec{v} = \vec{a} \cdot t, \vec{s} = \vec{a} \cdot \frac{t^2}{2}$ $\vec{a} = \overrightarrow{\text{constant}}$ $\dot{\vec{\omega}} = \overrightarrow{\text{constant}}$

$\vec{\omega} = \dot{\vec{\omega}} \cdot t, \vec{\phi} = \vec{\omega} \cdot \frac{t^2}{2}$

Momentum: $\vec{p} = m \cdot \vec{v}$

Kin. Energy: $E_{kin} = m \cdot \frac{v^2}{2}$

Momentum of rotation: $\vec{L} = I \cdot \vec{\omega}$

Rotat. Energy: $W_{rot} = I \cdot \frac{\omega^2}{2}$

Work and power:

$W = \int \vec{F} \cdot d\vec{s}$ $W = \int \vec{T} \cdot d\vec{\phi}$

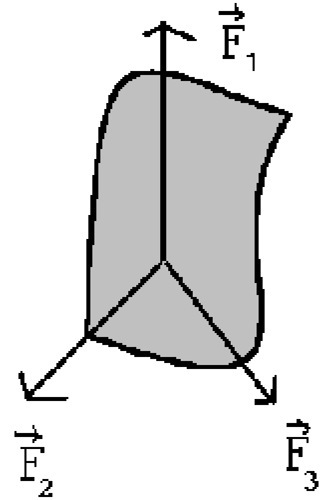
$P = \vec{F} \cdot \vec{v}$ $W = \vec{T} \cdot \vec{\omega}$

2.6. Equilibrium of a rigid body

Equilibrium: $\vec{F} = 0; \vec{T} = 0$

No translation: $\vec{F} = 0 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$| \vec{F}_3 | = | \vec{F}_2 | + | \vec{F}_1 |$$

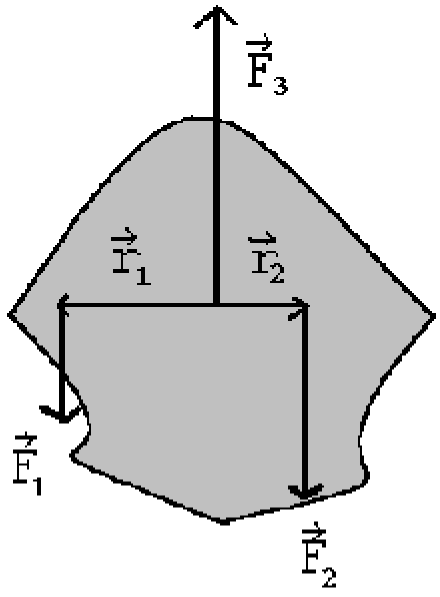


No rotation: $\vec{T} = 0; \vec{T}_1 + \vec{T}_2 = 0$

$$\vec{F}_1 \times \vec{r}_1 + \vec{F}_2 \times \vec{r}_2 = 0$$

$$F_1 \cdot r_1 = F_2 \cdot r_2$$

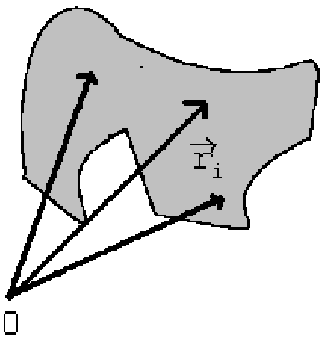
$$\Rightarrow \frac{F_1}{F_2} = \frac{r_2}{r_1}$$



Law of lever (Archimedes 250 v.Chr.)

Extended body:

Under what cirumstances is $\vec{T} = 0$



Torque due to gravity:

$$\vec{T}_g = \sum_i \vec{r}_i \times m_i \times \vec{g}$$

0 ? , if and only if $\sum_i m_i \cdot \vec{r}_i = 0$ $\vec{F}_i = m_i \cdot \vec{g};$

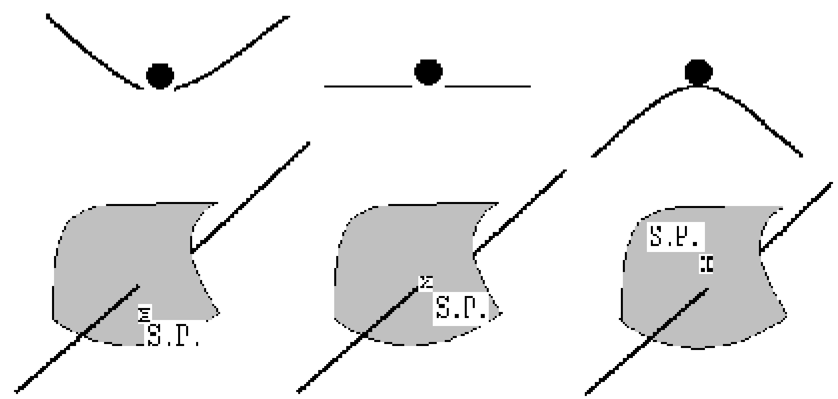
$$\vec{T} = 0$$

$$\vec{r}_{SP} = \frac{\sum_i m_i \cdot \vec{r}_i}{\sum_i m_i}$$

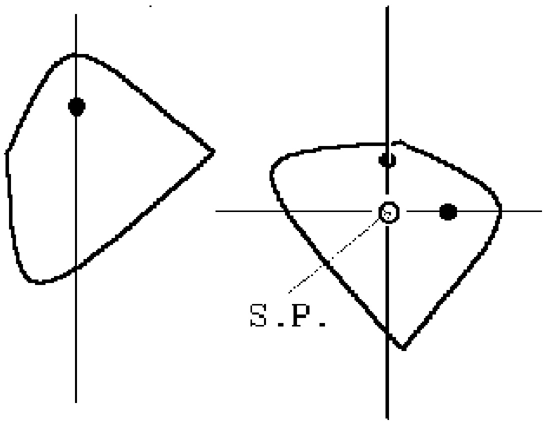
If origin = center of mass

States of equilibrium:

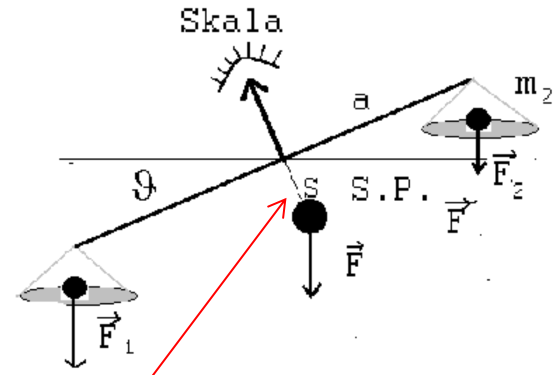
stable indifferent unstable



Experimental determination of the c.m. in a field of gravity



Balance:



$$\sum_i \vec{T}_i = 0$$

M: Total mass

$$m_1 + m_2 \neq M; m_1 - m_2 = \Delta M;$$

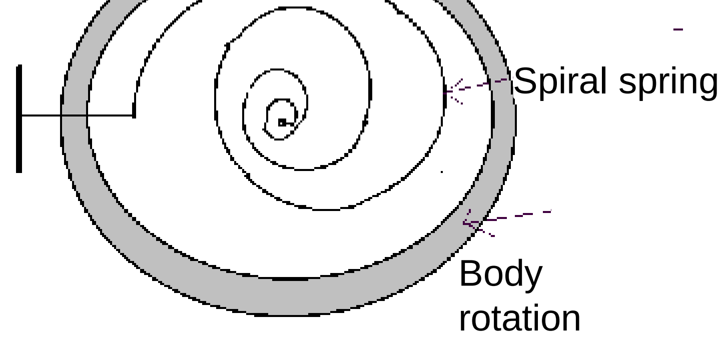
Sensibility of the balance:

ϑ : ',small" $\rightarrow \sin\vartheta \approx \vartheta$

$$\Rightarrow a \cdot \Delta M = s \cdot \vartheta \cdot g \cdot M \qquad a \cdot \cos\vartheta (m_1 - m_2) \cdot g = s \cdot \sin\vartheta \cdot g \cdot M$$

$$\frac{\vartheta}{\Delta M} = \frac{a}{s \cdot M} \Rightarrow$$ The smaller s, the larger the sensitivity

2.7. Rotational oscillations



At displacement: Torque $\approx \vartheta$

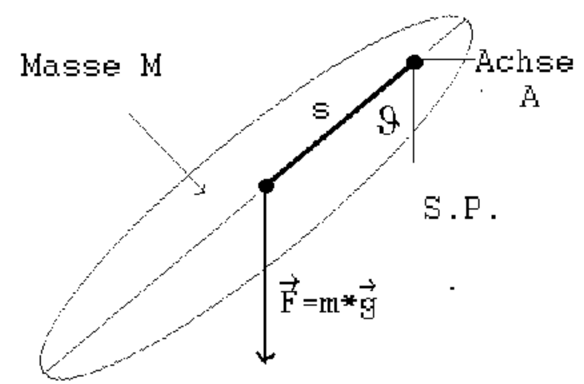
$\vec{T} = -D^* \cdot \vec{\vartheta};$ Angular benchmark

Equation of movement: $\boxed{I \cdot \ddot{\vartheta} = -D^* \cdot \vartheta}$

Solution: $\vartheta(t) = \vartheta_0 \cdot \cos\omega_0 \cdot t$ $\quad \vartheta_0$: max. displacement

$$\Rightarrow \omega_0 = \sqrt{\frac{D^*}{I}} \quad I = I_{sp} + M \cdot s^2$$

Compound pendulum



Torque: $-M \cdot g \cdot s \cdot \sin\vartheta$

$\sin\vartheta \approx \vartheta \quad \Rightarrow T = -M \cdot g \cdot s \cdot \vartheta$

$$I \cdot \dot{\omega} = I \cdot \ddot{\vartheta} \quad \Rightarrow I \cdot \ddot{\vartheta} = -M \cdot g \cdot s \cdot \vartheta$$

$\Rightarrow \omega_0 = \sqrt{\frac{M \cdot g \cdot s}{I_{sp} + M \cdot s^2}}$ mit $I = I_{sp} + M \cdot s^2$ $\quad \frac{s}{I_{sp} + M \cdot s^2} =$ Reduced length of the pendulum